SYMMETRIES OF AXION-DILATON STRING COSMOLOGY

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Abstract

The axion-dilaton string effective action is expressed in Einstein frame metric in a manifestly S-duality invariant form. It is shown that the moduli can be redefined to describe surface of a $(2 + 1)$-dimensional pseudosphere. The classical cosmological solutions of axion-dilaton are revisited. The Wheeler De Witt equation for the system is exactly solved and complete set of eigenfunctions are presented. The wave function factorizes and the one depending on the moduli is obtained by appealing to the underlying S-duality symmetry. Axion and dilaton parametrize $\frac{SU(1,1)}{U(1)}$, the S-duality group, and the Hamiltonian is expressed as a sum of the Ricci scalar and the Casimir of $SU(1,1)$. Therefore, the complete set of wave functions, depending on the moduli, is obtained from group theory technique. The evidence for the existence of a W-infinity algebra in axion-dilaton cosmology is presented and the origin of the algebra is primarily due to high degree of degeneracies in the wave functions. It is qualitatively argued that axion-dilaton quantum cosmology exhibits chaotic behavior in the semiclassical limit.

1 Introduction

It is well known that string theories possess rich symmetry structures. These symmetries have played central role in our understandings of various facets of string theories. Notable among the symmetries are dualities which have enabled us to explore string dynamics in diverse dimensions and have been guiding principles to seek an underlying fundamental theory in order to unify the five string theories [1]. The predictions of target space duality, T-duality, could be to tested in the frame work of perturbative string theory. On the other hand, the consequences of strong/weak coupling duality, the S-duality, are naturally nonperturbative in character.

One of the common attributes of the five perturbatively consistent string theories is that gravity is naturally incorporated in all of them and therefore, we can seek answers to deep questions in quantum gravity within the frame work of string theory. Indeed, string theory has provided deeper understandings about stringy black holes in the sense that the Bekenstein-Hawking entropy relation can be derived for a special class of black holes from the counting of microscopic states.

It is natural to explore the cosmological scenario from the perspectives of string theory. We expect that string theory will play important role in studying the evolution of the Universe in early epochs and shed lights on the mechanism of inflation. It is hoped that string theory will provide resolution of the initial singularity problem in cosmology. There have been considerable amount of activity in order to investigate various aspects of string cosmology during the past decade [2, 3]. In this context, symmetries too have played a key role in formulating stringy cosmological models. The pre-big bang (PBB) [4] proposal has received a lot of attention since it incorporates stringy symmetries in order to provide a novel mechanism for inflation.

It is worth while to recall the role of dilaton, ϕ, in string theory. The dilaton appears as a massless excitation in string theory along with other massless excitations of the spectrum. An important point is that the exponential of the VEV of dilaton is the coupling constant in string theory and in turn it determines the Newton's constant, the gauge coupling constants and Yukawa couplings of low energy effective actions derived to describe models of particle physics. Moreover, when one envisages cosmological situation, the dilaton naturally appears in the effective action along with graviton. In fact the graviton-dilaton pair play a crucial role for the existence of scale factor duality (SFD) [5] in string cosmology and the analog of this symmetry does not exist in pure Einstein gravity. Moreover, it is well known that the the pair dilaton, ϕ, and axion, χ, parametrize the S-duality group [6, 7] $\frac{SL(2,R)}{U(1)} \sim \frac{SU(1,1)}{U(1)}$. The axion, as a weakly coupled pseudoscalar boson, is incorporated in the standard model of particle physics and is expected to play a vital role in cosmology. Therefore, it is interesting to explore various aspects of axion-dilaton cosmology in the string theory.

This article is a continuation of our study of classical and quantum aspects of axion-dilaton string cosmology. Recently, we have presented some of our preliminary results in a short communication [8]. We briefly discuss known classical solutions for the sys-

tem and explore further the quantum cosmological solutions. The S-duality symmetry plays a central role in our investigation. At the classical level, we construct the generators of the symmetry and present the algebra of the generators. The problem may be conveniently studied by a suitable change of variable where the axion dilaton moduli space is described as the surface of a $2 + 1$ dimensional pseudosphere. In the cosmological scenario, the moduli depend on cosmic time coordinate and the classical Hamiltonian constraint reflects invariance of the action under time reparametrization. In quantum cosmology, the Hamiltonian constraint translates to an operator condition, the Wheeler De Witt equation [9].

We present a class of solutions to WDW equation for the case at hand. These solutions are new, were missed in earlier investigations [10, 11] and are obtained from purely group theoretic considerations for the moduli describing surface of the pseudosphere. It is found that the wave function is highly degenerate. Furthermore, we explore the possibility of solving WDW equation for the $\chi - \phi$ system directly. It is shown that, with a well known redefinition of these fields, the problem can be mapped to that of motion on the Poincaré upper half plane. The classical problem, in the absence of coupling to gravity has been studied quite well. The solution to classical equations of motion, for axion-dilaton system coupled to gravity, are well known. In the context of quantum mechanics, it turns out that separation of variable for the wave function of the Universe is possible when we write down the WDW equation. Then one has to solve a differential equation involving the scale factor and another one which is the analog of Schrödinger equation in the Poincaré upper half plane. Since the problem of motion on surface of a pseudosphere and that of Poincaré upper half plane are related [12], it is not surprising that the wave function derived in the latter formulation is also highly degenerate. One of the main difference in the characteristics of the wave function derived for motion of pseudosphere is that we seek a simultaneous eigenfunction of the Casimir operator and the only compact generator of $SU(1, 1)$. In case of the Poincaré upper half plane, we choose solutions such that the wave function can be expressed as a product of a plane wave in the axion field and a function of the dilaton. Obviously, the plane wave part has continuous eigenvalue and is related to an eigenfunction of a noncompact generator of the S-duality group. There is some advantage in solving the quantum equation in terms of coordinates defining the Poincaŕe upper half plane. Here, we can directly study the behaviour of the wave function in the strong coupling limit which is not so transparent when we solve WDW equation in terms of the coordinates of the pseudosphere. On the other hand, some of the symmetry properties are more directly brought out in formulating the problem with coordinates of the pseudosphere.

The WDW equation is a second order differential which will have infinite number of solutions in general. Therefore, it is necessary to supplement the solutions with boundary conditions. When we seek solutions to wave equations in quantum mechanics, the boundary conditions are imposed by taking physical considerations into account from out side. In contrast, when one derives the wave function of the Uni-

verse, the boundary conditions are generally considered as additional principles. The well known boundary conditions are the Hartle-Hawking no boundary proposal [13] and the tunneling scenario advanced by Linde and Vilenkin [14, 15].

In the context of string cosmology, it is necessary to envisage the boundary conditions in an appropriate perspective. Here, dilaton always accompanies graviton and it is also the coupling constant as alluded to earlier. Thus, when the boundary condition on the wave function of the Universe is considered, this aspect should be kept in mind. In the PBB scenario, the classical solution is such that initially, the Universe evolves from low curvature, low temperature phase and is in weak the coupling regime. Subsequently, it undergoes accelerated expansion. When we consider quantum string cosmological equation in the frame work of PBB proposal, the boundary condition on the wave function is not introduced from outside [3, 16]. The PBB mechanism already dictates which boundary condition is to be respected by the wave function. Furthermore, in the string frame, the Hamiltonian manifestly is invariant under the scale factor duality and therefore, the quantum Hamiltonian is determined so as to respect this symmetry. Note that in this case the string symmetries impose additional conditions on the wave functions of WDW equation [16]. Therefore, one might contemplate that a third of boundary condition is available as an option in quantum string cosmology.

There is a general trend in string theory (in the sense of string effective action) that as we envisage theories in lower spacetime dimensions, there is enhancement of symmetries. Let us very briefly recapitulate some of the important features of string effective action in this perspective. We may recall a generic feature of the toroidal compactification of string effective action from critical, 10, dimension to lower dimension. If we consider heterotic string as an illustrative example, we note that the four dimensional theory possesses, not only the T-duality group $O(6, 22)$, but also it is endowed with S-duality symmetry (at the level of equations of motion). When one compactifies the theory on seven torus to go to three dimensions the group in $O(8, 24)$ which contains both T-duality group, $O(7, 23)$ as well as the S-duality group, $SL(2, R)$ [17]. Furthermore, the the theory in two spacetime dimensions is known to admit affine algebras [18]. Thus, it might be interesting to explore whether the theory depending on one coordinate, in the present context the cosmological scenario, contains an underlying symmetry other than the S-duality which is manifest in the problem under consideration. We have discovered a w_∞ algebra whose generators are constructed from the S-duality group, $SU(1, 1) \sim SL(2, R)$ and they act on the space of solutions of the WDW equation [8].

The plan of the paper is as follows: In Section 2, we present the tree level string effective action for graviton, axion and dilaton in the Einstein frame. We rewrite the action in a manifestly S-duality invariant form and the we identify the group to be $SU(1, 1)$ and discuss the invariance properties of the action. The conserved currents, associated with the symmetry transformations, are constructed from the standard Nöther procedure. We study the cosmological scenario in Section 3. First, the classi-

cal cosmological axion-dilaton solutions are briefly discussed. Next, the Hamiltonian constraint is derived and the Wheeler De Witt equation is obtained. It is argued that the wave function could be expressed in the factorized form separating into function of the scale factor a and that of the moduli. The WDW equation is first solved in the Lorentzian coordinates representation of $(2+1)$-dimensional the pseudosphere. The eigenfunctions for the moduli are presented from the group theoretic considerations and their properties are elucidated. Next, wave function depending the scale factor is obtained. The solutions to WDW equation is also given when the moduli is expressed in the Poincaré coordinates defining the moduli space to be the upper half plane. This choice of coordinate system has the advantage that the strong coupling limit of the wave function is rather transparent and it is easy to exhibit the semiclassical limit of the theory in simple way. Section 4 contains discussion of the W-infinity algebras in axion-dilaton cosmology which is a more detailed analysis of our preliminary conjecture [8]. In Section 5, we qualitatively explore the phenomena of chaos in axion-dilaton quantum cosmology. The last Section is devoted to summary and discussion of our results. The Appendix contains some relevant formulas for the $SU(1,1)$ group.

2 Effective Action and S-duality Symmetry

Let us consider a four dimensional string effective action in the presence of axion and dilaton.

$$S_4 = \int d^4x \sqrt{-g} \left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 \right) \tag{1}$$

Here R is the scalar curvature, computed from the Einstein frame metric $g_{\mu\nu}$, $\sqrt{-g}$ is its determinant and the other two terms correspond to kinetic energy terms of dilaton and axion respectively. This action may be thought of as the one derived from ten dimensional type IIB effective [19] action where rest of the background fields are set to zero. Alternatively, we could derive this action from NS-NS sector of a string theory where the field strength of $B_{\mu\nu}$, $H^{\mu\nu\lambda} = e^{2\phi}\epsilon^{\mu\nu\lambda\rho}\partial_\rho\chi$, is dualized to introduce the axion. It is more convenient to express the action in a manifestly S-duality invariant form as given below

$$S_4 = \int d^4x \sqrt{-g} \left(R + \frac{1}{4}\text{Tr}[\partial_\mu \mathbf{V}^{-1}(x)\partial^\mu \mathbf{V}(x)] \right) \tag{2}$$

where the 2×2 matrix $\mathbf{V}$ is given by

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} A+B & 2B\chi + i(A-B) \\ 2B\chi - i(A-B) & A+B \end{pmatrix} \tag{3}$$

The elements of the 2×2 matrix $\mathbf{V}$ are defines as: $A = e^{-\phi} + \chi^2 e^\phi$ and $B = e^\phi$. Note that $\mathbf{V} \in SU(1,1)$ and satisfies $\mathbf{V}^{-1} = \sigma_3 \mathbf{V}^\dagger \sigma_3$ and $\sigma_i, i = 1, 2, 3$ are the Pauli

matrices. The action (2) is invariant under

$$\mathbf{V} \to \Omega^\dagger \mathbf{V} \Omega, \quad g_{\mu\nu} \to g_{\mu\nu} \quad \text{and} \quad \Omega^\dagger \sigma_3 \Omega = \sigma_3, \tag{4}$$

where $\Omega \in SU(1,1)$ and σ_3 is the metric which is required to be invariant under the group transformations. We recall that any 2×2 matrix $\mathbf{U} \in SU(1,1)$, in general is be spacetime dependent, can be expressed as

$$\mathbf{U} = \mathbf{1}U_0 + \sigma_1 U_1 + \sigma_2 U_2 + i\sigma_3 U_3 \tag{5}$$

and the spacetime dependent coefficients satisfy the constraint

$$U_0^2 + U_3^2 - U_1^2 - U_2^2 = 1 \tag{6}$$

Note from the structure of the $\mathbf{V}$ matrix (3) that, if we expand it in terms of unit matrix and σ_i, the coefficient of σ_3 turns out to be zero; therefore, the constraint satisfied by the coefficients v_i, $i = 0, 1, 2$ is

$$v_0^2 - v_1^2 - v_2^2 = 1 \tag{7}$$

Therefore, the moduli define surface of a $(2 + 1)$-dimensional psuedosphere as mentioned earlier. We can read off the coefficients from (3)

$$v_0 = \frac{1}{2}(A + B), \quad v_1 = B\chi, \quad v_2 = \frac{1}{2}(B - A) \tag{8}$$

and it is easy to check that they satisfy the requisite constraint (7).
We may write the action (2) in terms of the moduli as

$$S = \int d^4x \sqrt{-g} \left(R + \frac{1}{2} g^{\mu\nu} \eta^{ij} \partial_\mu v_i \partial_\nu v_j \right) \tag{9}$$

The metric in this moduli space is $\eta^{ij} = \text{diag}(1, -1, -1)$ with $i, j = 0, 1, 2$ and the constraint (7) reads $\eta^{ij} v_i v_j = 1$.
The generators of $SU(1,1)$, denoted by J_1, J_2, and J_3 satisfy the commutation relation

$$[J_1, J_2] = -iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2 \tag{10}$$

The set of 2×2 matrices satisfying the $SU(1,1)$ algebra are

$$J_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad J_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

As is well known, for a noncompact group like $SU(1,1)$, it is not possible to choose all generators as 2×2 Hermitian matrices. Now let us consider an infinitesimal $SU(1,1)$ transformations on the moduli v_i. We can express an $SU(1,1)$ transformation as

$$\Omega = e^{i\alpha_k J_k} \sim 1 + i\alpha_k J_k \tag{12}$$

5

for the infinitesimal parameters $\{\alpha_i\}$. Thus the variation of the $\mathbf{V}$ matrix $\delta\mathbf{V} = \mathbf{\Omega}^\dagger\mathbf{V}\mathbf{\Omega} - \mathbf{V}$ is expressed as

$$\delta\mathbf{V} = \mathbf{1}\delta v_0 + \sigma_1\delta v_1 + \sigma_2\delta v_2 \tag{13}$$

and the variations of v_i are given by

$$\delta v_0 = -(\alpha_1 v_1 + \alpha_2 v_2), \quad \delta v_1 = -(\alpha_1 v_0 + \alpha_3 v_2), \quad \delta v_2 = \alpha_3 v_1 - \alpha_2 v_0 \tag{14}$$

We remind the reader that the Einstein frame metric remains invariant under S-duality transformation and it is rather straight forward to construct the three conserved currents from the action (9).

$$\mathcal{J}_1^\mu = \sqrt{-g}(\partial^\mu v_1 v_0 - \partial^\mu v_0 v_1) \tag{15}$$

$$\mathcal{J}_2^\mu = \sqrt{-g}(\partial^\mu v_2 v_0 - \partial^\mu v_0 v_2) \tag{16}$$

$$\mathcal{J}_3^\mu = \sqrt{-g}(\partial^\mu v_1 v_2 - \partial^\mu v_2 v_1) \tag{17}$$

The charge densities can be expressed as

$$\mathcal{Q}_1 = (p_1 v_0 + p_0 v_1), \quad \mathcal{Q}_2 = (p_2 v_0 + p_0 v_2), \quad \mathcal{Q}_3 = (p_1 v_2 - p_2 v_1) \tag{18}$$

where the canonical momenta are defined from the Lagrangian given by (9). We mention in passing that the generators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are noncompact and they are like Lorentz generators on the pseudosphere; whereas, $\mathcal{Q}_3$ is compact and generates rotation on $v_1 - v_2$ plane. The charge densities satisfy $SU(1,1)$ algebra (10) when we compute their Poisson brackets and the factor of 'i' is recovered when the naive commutation relations are carried out ignoring the subtleties that arise in product of operators.
It is also useful to explore how the $SU(1,1)$ algebra emerges starting from action (1). Let us define a complex scalar field

$$\lambda = \lambda_1 + i\lambda_2, \quad \text{where} \quad \lambda_1 = \chi \quad \text{and} \quad \lambda_2 = e^{-\phi} \tag{19}$$

Then (1) is expressed as

$$S_\lambda = \int d^4x\sqrt{-g}\left(R - \frac{1}{2\lambda_2{}^2}g^{\mu\nu}\left[\partial_\mu\lambda_1\partial_\nu\lambda_1 + \partial_\mu\lambda_2\partial_\nu\lambda_2\right]\right) \tag{20}$$

Let us consider following set of transformations which leave the Einstein metric invariant and act only on λ_1 and λ_2.
(i) Dilation: $\delta\lambda_1 = \epsilon_1\lambda_1, \quad \delta\lambda_2 = \epsilon_1\lambda_2$.
(ii) Translation along λ_1: $\delta\lambda_1 = \epsilon_2, \delta\lambda_2 = 0$.
(iii) Nonlinear rotation: $\delta\lambda_1 = \epsilon_3(\lambda_2^2 - \lambda_1^2), \quad \text{and} \quad \delta\lambda_2 = -2\epsilon_3\lambda_1\lambda_2$.
Where $\epsilon_i, \ i = 1,2,3$ are constant parameters. We may derive the conserved currents, following Nöther's prescriptions. The charge densities are given by

$$Q_1 = \lambda_1 p_{\lambda_1} + \lambda_2 p_{\lambda_2}, \quad Q_2 = p_{\lambda_1}, \quad Q_3 = p_{\lambda_1}(\lambda_2^2 - \lambda_1^2) - 2\lambda_1\lambda_2 p_{\lambda_2} \tag{21}$$

It is easy to check that the Poisson bracket algebra of three charges obtained by the spatial volume integration of the densities Q_i close on to algebra of the generators of the S-duality group. The following linear combination of the three charges satisfy the algebra of generators given in (10)

$$J_1 = \frac{1}{2}(q_1 + q_2), \quad J_2 = q_1 \;\; \text{and} \;\; J_3 = \frac{1}{2}(q_2 - q_3) \tag{22}$$

where the charges, $q_i = \int dV Q_i$.

3 The Cosmological Scenario

We presented the symmetries of the four dimensional effective action with axion and dilaton as spacetime dependent matter fields. The generators of the S-duality group, $SU(1,1)$ were constructed in terms of the axion and dilaton fields as well as in terms of the new fields v_i which describe surface of the pseudusphere. In the cosmological case, the metric and the matter fields are assumed to depend on cosmic time. Furthermore, we adopt homogeneous, isotropic FRW metric

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \tag{23}$$

$a(t)$ is the scale factor and t is cosmic time. The constant, k, takes values $k = +1, 0, -1$ which corresponds to closed, flat and open Universes respectively. The Einstein field equation, used to solve for the scale factor, is given by

$$R_0^0 - \frac{1}{2}R = {}^{(\phi)} T_0^0 + {}^{(\chi)} T_0^0 \tag{24}$$

where R_0^0 is the $0-0$ component of the Ricci tensor and the right hand side is the sum of the $0-0$ components of stress energy momentum tensor for dilaton and axion as is obvious from the notation. The action, in minisuperspace, is given by

$$S = \int dt(-6a\dot{a}^2 + 6ka + \frac{1}{2}a^3\dot{\phi}^2 + \frac{1}{2}e^{2\phi}a^3\dot{\chi}^2) \tag{25}$$

This action is expressed in terms of the Einstein frame metric. In the above equation, we have not explicitly written the total derivative term which arises when we partially integrate the piece coming from the scalar curvature. As a simple, illustrative example, let us look at the classical solutions of field equations for the case $k = 0$. The other two equations corresponding to axion and dilaton evolutions are

$$\ddot{\chi} + 3H\dot{\chi} + 2\dot{\phi}\dot{\chi} = 0 \tag{26}$$

7

$$\ddot{\phi} + 3H\dot{\phi} = e^{2\phi}\dot{\chi}^2 \qquad (27)$$

where $H = \frac{\dot{a}}{a}$, is the Hubble parameter. Notice that the axion equation leads to a charge conservation law since

$$\frac{d}{dt}\left(ln\ \dot{\chi} + lna^3 + 2\phi\right) = 0 \qquad (28)$$

which implies $a^3\dot{\chi}e^{2\phi} = \pm L$, L being a constant of motion. The two time evolution equations for axion and dilaton (26) and (27) may be solved more efficiently [20] if one introduces a line element

$$d\xi^2 = d\phi^2 + e^{2\phi}d\chi^2 \qquad (29)$$

leading to equation of motion

$$\ddot{\xi} + 3H\dot{\xi} = 0 \qquad (30)$$

Again leading to a conservation law $a^3\dot{\xi} = \pm K$, where K is a positive constant. Then one could solve the Einstein-Friedmann equation by utilizing the conservation law of the ξ variable. It suffices for our purpose to note that

$$a \sim t^{\frac{1}{3}}, \quad e^\phi \sim \frac{L}{2K}\left((\frac{t}{t_0})^{-\frac{2}{\sqrt{3}}} + (\frac{t_0}{t})^{-\frac{2}{\sqrt{3}}}\right) \ \text{and} \ \chi \sim \text{Const}\frac{K}{L}\frac{1}{1 + (\frac{t}{t_0})^{-\frac{4}{\sqrt{3}}}} \qquad (31)$$

Note that the solutions presented here are in terms of cosmic time, t and t_0 is a constant of integration, In ref. [20] the solutions were obtained in conformal time frame. Let us dwell on some of the features of the solution (31). We note that the scale factor $a(t) \to 0$ as $t \to 0$ which corresponds to curvature singularity. At this point dilaton diverges; in other words, as $t \to 0$ the coupling constant becomes strong. However, the axion tends to a constant value in the $t \to 0$ limit. As is well known, it is not possible to avoid the curvature singularity by a different choice of coordinate frame. The classical solution of axion-dilaton system has been studies in detail in [20] for the cases $k = \pm 1, 0$ and it was found that dilaton approaches strong coupling limit in all these cases. Note that in absence of the axion (i.e. K=0), the solution is obtained for graviton-dilaton system in the Einstein frame, where we see the presence of two solutions for the scale factor and also there are two different solutions for the dilaton.

Now we proceed to analyze quantum axion-dilaton cosmology in more detail. In order to set up the Wheeler De Witt equation with manifest S-duality symmetry and to facilitate the solutions of the quantum mechanical equations, it is more convenient to deal with the moduli $\{v_i\}$. Note that the scalar curvature is given by i.e. $\sqrt{-g}R = 6(-a\dot{a}^2 + ka)$ for our choice of FRW metric and we have dropped a total derivative term here. The canonical momentum associated with the scale factor becomes $P_a = -12\dot{a}a$. Consequently, when we obtain the canonical Hamiltonian there will be unconventional numerical factor like $\frac{1}{24}$ factor multiplying P_a^2 which are inconvenient to keep track

of. Therefore, as is the accepted prescription, we rescale the metric and the moduli v_i accordingly so that the cosmological action takes the following form

$$\tilde{S} = \frac{1}{2} \int dt \left(-a\dot{a}^2 + ka - a^3 \dot{v}_i \dot{v}_j \eta^{ij} \right) \tag{32}$$

Let us consider $k = 1$ as a generic case from now on; $k = 0$ will be rather simple and $k = -1$ can be dealt with as one solves $k = +1$ situation. We shall see that the solution to the 'angular part' of the WDW equation involving axion and dilaton is not affected by the choice of k. This part of the WDW wave function will be determined from purely group theoretic considerations. Therefore, some of our general conclusions are not affected by this choice of k. First, we derive the Hamiltonian constraint corresponding to (32)

$$\mathcal{H} = \frac{1}{2} \left(\frac{1}{a} P_a^2 + a + \frac{1}{a^3} \eta_{ij} P_v^i P_v^j \right) = 0. \tag{33}$$

where $\mathcal{H}$ is the canonical Hamiltonian derived from (32) and the canonical momenta are defined as

$$P_a = \frac{\partial L}{\partial \dot{a}}, \quad P^i = \frac{\partial L}{\partial \dot{v}_i} \tag{34}$$

We remark in passing that the action (32) is invariant under the $SU(1,1)$ transformations and so is the Hamiltonian (33). It is easy to check that the Hamiltonian commutes with the generators of $SU(1,1)$ as well as with the Casimir.

When we implement (33) as a quantum constraint, $\mathcal{H}$ is replaced by operator, $\hat{\mathcal{H}}$, which acting on the wave function is required to give zero eigenvalue. We identify $\hat{P}_a = -i\frac{\partial}{\partial a}$ and $\hat{P}^i = -i\frac{\partial}{\partial v_i}$. Note that the first term in the expression for the classical (33) contains a product of the scale fact, a, and its canonical momentum. In defining the quantum Hamiltonian, we encounter the well known ordering ambiguity since product of two noncommuting operators appears. We adopt the the prescription such that the resulting Hamiltonian respects invariance under change of coordinates in minisuperspace [21]. The next important point to note that the last term in (33) is related to the Casimir operator of $SU(1,1)$ and therefore $\mathcal{H}$ is S-duality invariant. We can recognize this if we consider 'polar' coordinates

$$v_0 = \cosh\alpha, \quad v_1 = \sinh\alpha\cos\beta \text{ and } v_2 = \sinh\alpha\sin\beta \tag{35}$$

These variables are also called 'disc variables'; where α is real and $0 \le \beta \le 4\pi$, since the magnetic quantum number takes both integer and half integer values as we shall see later. Furthermore, axion and dilaton can be expressed as

$$\chi = \frac{\sinh\alpha\cos\beta}{(\cosh\alpha + \sinh\alpha\sin\beta)}, \quad e^{-\phi} = \frac{1}{(\cosh\alpha + \sinh\alpha\sin\beta)} \tag{36}$$

The Casimir operator, in terms of the 'polar' coordinate system is just the Laplace-Beltrami operator given by

$$\hat{C} = -\frac{1}{\sinh\alpha} \frac{\partial}{\partial\alpha} \sinh\alpha \frac{\partial}{\partial\alpha} - \frac{1}{\sinh^2\alpha} \frac{\partial^2}{\partial\beta^2} \tag{37}$$

9

If we denote eigenvalue of the Casimir operator, $\hat{C}$, by C, then the quantum Hamiltonian satisfies WDW equation i.e. $\hat{\mathcal{H}}\Psi = 0$ which assumes the following form as a differential equation

$$\left(\frac{\partial^2}{\partial a^2} + \frac{\partial}{\partial a} - a^2 + \frac{1}{a^2}\hat{C}\right)\Psi = 0. \tag{38}$$

We may express Ψ as product of two functions: one depending on the scale factor, a and other depending on the two 'polar' coordinates: $\Psi = \mathcal{U}(a)Y$. Here Y is the eigenfunction of the Casimir operator $\hat{C}Y = j(j+1)Y$. We may choose the eigenfunction to be also simultaneous eigenfunction one one of the generators of $SU(1,1)$ and here we choose that operator to the compact generator, J_3. Thus the goal is to solve for the relations

$$\hat{C}|j, m> = j(j+1)|j, m>, \text{ and } J_3|j, m> = m|j, m> \tag{39}$$

Now the problem is reduced to identifying all unitary, infinite dimensional representations of $SU(1,1)$ satisfying above requirements. We present below the classifications of the representations of $SU(1,1)$ which are well known [22, 23, 24] and refer to readers to the Appendix for some more relevant details.

The representations of $SU(1,1)$ mainly fall into three different categories:

(i) The discrete series $D_j^{\pm}$.

D_j^+ is the one for which $j = -\frac{1}{2}, -1, -\frac{3}{2}, -2....$ For a given j value, the eigenvalue of J_3, m, is unbounded from above taking values $m = -j, -j+1, -j+2, ...$

On the other hand, for the other discrete series D_j^-, j is negative, taking integer and half integer values. Moreover, in this case, m is unbounded from below and therefore, $m = j, j-1, j-2, ...$ We adopt the convention where j *is negative*.

The two sets of wave functions are related by a symmetry when $m \to -m$ which follows from the properties of the $D_{mm'}^j$ functions of $SU(1,1)$ group (see the Appendix for definition of these functions).

(ii) The continuous series are also of two types; C_l^0 and $C_l^{\frac{1}{2}}$ and j turns out to be complex in this case:

$$j = -\frac{1}{2} + il, \quad l > 0, \quad \text{and real} \tag{40}$$

For C_l^0 the eigenvalues of J_3 take positive and negative integer value for each j i.e. $m = 0, \pm 1, \pm 2, ...$ On the other hand, $m = \pm\frac{1}{2}, \pm\frac{3}{2}, ...$ for the $C_l^{\frac{1}{2}}$ series. Note that $j^* = -j - 1$ and also that the eigenvalue of the Casimir operator is real.

(iii) The supplementary series is defines when the value of j lies in the range $-\frac{1}{2} < j < 0$ and the magnetic quantum number takes integer values in this case: $m = 0, \pm 1, \pm 2, ...$ There is a theorem due to Bargmann which states that any function defined on $SU(1,1)$ can be expanded in terms of the set of functions belonging to discrete and continuous series $\{D_j^{\pm}, C_l^O, C_l^{\frac{1}{2}}\}$ and it is not necessary to include the

functions belonging to the supplementary series.

At this stage, it is worth while to discuss a few points before we derive the explicit form of the wave function. The constrain equation for v_i's i.e. $v_0^2 - v_i^2 - v_2^2 = 1$; describes, as mentioned earlier, a surface of constant negative curvature in the moduli space. The constraint equation is for the moduli and v_0 has nothing to do, we emphasize, with spacetime coordinates at all. It is just one of the three moduli. The equation defining the pseudosphere has interesting geometrical structure and let us recall some of these features [12]. The locus of points equidistant of the origin specifies a hyperboloid of two sheets intersecting the v_0-axis at the point $v_0 = \pm 1$. These two points, in analogy with the S^2 are called poles. Due to the nature of this geometry, each sheet models an infinite space-like surface and this surface has no boundary. Moreover, the geodesics are intersection of hyperboloid sheet with planes through the origin.

Another comment is in order here. We have expressed the wave function of the WDW equation (38) in the factorized form and thus one of our goal is to solve the wave equation on the pseudosphere. It is well known from earlier studies [27] that the solution admits continuous values for eigenvalue of the Casimir operator and we shall give heuristic physical argument below to support this claim. In other words, although the unitary infinite dimensional representations of $SU(1,1)$ admit both discrete and continuous values for j (and hence the Casimir), for the case at hand we only need to pick up the solutions corresponding to the continuous one. For convenience, introduce the continuous eigenvalue ρ which appears in eigenvalue equation as follows

$$\hat{C}Y_\rho^m = (\rho^2 + \frac{1}{4})Y_\rho^m, \quad J_3 Y_\rho^m = m Y_\rho^m \tag{41}$$

There is an intuitive way to see why the continuous eigenvalue solutions are the correct ones. For notational simplicity, momentarily, let us write (37) as

$$O_{LB}\psi = \epsilon\psi \tag{42}$$

O_{LB} is the Laplace-Beltrami operator defined in (37) and $\epsilon = \rho^2 + \frac{1}{4}$ is the eigenvalue. Note that ρ is responsible for exponential behaviour of the wave function at infinity. The disc boundary corresponds to $\alpha \to \infty$. If we want to see how ψ behaves on the boundary, then (42) could be well approximated (note: $\sinh\alpha \sim \frac{e^\alpha}{2}$) by

$$-\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial}{\partial\alpha}\right)\psi = \epsilon\psi \tag{43}$$

Therefore, near the disc boundary

$$\psi \sim e^{-\alpha/2}e^{\pm i\rho\alpha}g(\alpha) \tag{44}$$

We have displayed the exponential α-dependence in two terms explicitly and $g(\alpha)$ is some smooth function. Also note that the probability density in our curved space is $|\psi|^2\sinh\alpha d\alpha d\beta$ and this explains the first exponential. When $\epsilon \geq \frac{1}{4}$, ρ is real and

11

second exponential is like a plane wave part. Let us assume that we admit eigenvalues $\epsilon < \frac{1}{4}$, in that case ρ becomes imaginary. Consequently, the wave function will grow exponentially in some direction. If it falls off exponentially in all directions, we may conclude that this solution corresponds to a bound state in a homogeneous space where no interaction potential is present. On the other hand, the exponentially growing solutions, alluded to earlier (still $\epsilon < \frac{1}{4}$ is being discussed), are not admissible as wave functions of any continuous spectra. We are led to conclude, from these reasonings, that the eigenvalues of O_{LB} on the pseudosphere, as being studied here, are continuous and the lowest eigenvalue, $\epsilon_0 = +\frac{1}{4}$; and furthermore, the eigenvalues extend up to ∞.

Now we return to discuss the eigenfunctions of $\hat{C}$ and present explicit solutions of the wave function. We choose m to take integer values. Recall that the form of the differential operator $\hat{C}$ is given by (37) and the compact operator $J_3 = -i\frac{\partial}{\partial\beta}$, then we can write

$$Y_\rho^m = e^{im\beta} X_\rho^m \tag{45}$$

Here of course β takes values 0 to 2π. The differential equation satisfied by X_ρ^m is the familiar one

$$-\frac{1}{\sinh\alpha}\frac{\partial}{\partial\alpha}(\sinh\alpha)\frac{\partial}{\partial\alpha}X_\rho^m + \frac{m^2}{\sinh^2\alpha}X_\rho^m = (\rho + \frac{1}{4})X_\rho^m \tag{46}$$

The solutions are associated Legendre functions [25, 28] $P_{-\frac{1}{2}+i\rho}^m$. Thus, the normalized eigenfunction is

$$Y_\rho^m(\alpha, \beta) = \frac{e^{im\beta}}{\sqrt{2\pi}}\frac{\Gamma(i\rho + \frac{1}{2} - m)}{\Gamma(i\rho)}P_{-\frac{1}{2}+i\rho}^m(\cosh\alpha) \tag{47}$$

Notice that the wave functions thus obtained belong to the continuous series of the representations of $SU(1, 1)$. We present below the representation of the associated Legendre functions in terms of hypergeometric functions

$$P_l^m(w) = \frac{1}{\Gamma(1-m)}(\frac{w+1}{w-1})^{\frac{m}{2}} {}_2F_1(-l, l+1; 1-m; \frac{1-w}{2}) \tag{48}$$

where $l = -\frac{1}{2} + i\rho$. Moreover, the canonical function $P_{-\frac{1}{2}+\rho}^m(w)$, $m \in \mathbf{Z}$, have special properties

$$P_{-\frac{1}{2}+i\rho}^m(w) = P_{-\frac{1}{2}-i\rho}^m(w) \tag{49}$$

which, we may recall, follows from the relation between P_l^m and P_{-l-1}^m. Since $j = -\frac{1}{2} + il$, $l > 0$ for the continuous series, one may utilize this property the establish the orthogonality relation

$$|\frac{\Gamma(\frac{1}{2} + i\rho - m)}{\Gamma(i\rho)}|^2 \int_1^{infty} P_{-\frac{1}{2}+i\rho}^m(w)P_{-\frac{1}{2}-i\rho}^m(w)dw = \delta(\rho - \rho') \tag{50}$$

12

The orthogonality relation is defined for $\rho, \rho' \in \mathbf{R}^+$ and $m \in \mathbf{R}$. The completeness relation reads as

$$\int_0^\infty |\frac{\Gamma(\frac{1}{2} + i\rho - m)}{\Gamma(i\rho)}|^2 P_{-\frac{1}{2}+i\rho}^m(w) P_{-\frac{1}{2}-i\rho}^m(w') d\rho = \delta(w - w') \tag{51}$$

These relations are known as Mehler transformations [25]. Now, using the value of the Casimir in (38), we arrive at the differential equation

$$\left(\frac{d^2}{da^2} + \frac{d}{da} - a^2 + \frac{\rho^2 + \frac{1}{4}}{a^2} \right) \mathcal{U}(a) = 0 \tag{52}$$

The solutions are Bessel functions [26]: $\mathcal{U}(a)_\nu = J_{\pm i\nu/2}(\frac{i}{2}a^2)$ where $\nu^2 = \rho^2 + \frac{1}{4}$ is introduced for notational conveniences. Thus, the solution, Ψ, is given by

$$\Psi(a, \alpha, \beta) = \mathcal{U}(a)_\nu e^{im\beta} P_{-\frac{1}{2}+i\rho}^m(\cosh\alpha) \tag{53}$$

Let us discuss the necessary inputs for determining the wave function from the available class of solutions. The a dependent part of the wave function admits two possibilities, in $a \geq 0$ region, where the wave function corresponds to classical configurations such that either the Universe is expanding or it is contracting. We should choose the solution which is identified with an expanding Universe. Moreover, the wave function is expected to be well behaves for large a. The first criterion (expanding Universe i.e. Hubble parameter $H > 0$) implies that in the limit $a \to 0$ the wave function should have negative eigenvalue of the momentum $\hat{P}_a$, since classically $H = -\frac{P_a}{a^2}$. The function that fulfills this requirement is $H_{\frac{i}{2}\nu}^{(1)}(z)$, where $z = \frac{i}{2}a^2 = \frac{e^{\frac{i\pi}{2}}}{2}a^2$. In the limit $a \to 0$

$$H_{\frac{i}{2}\nu}^{(1)} \to \frac{1}{\pi}\Gamma(\frac{i}{2}\nu)e^{\frac{\pi}{4}}e^{-i\gamma} \tag{54}$$

where $\gamma = lna$ introduced for brevity of notation. Note that $\dot{\gamma} = H = -\frac{P_a}{a^2}$ and $P_\gamma = -\frac{3}{2}e^{3\gamma}\dot{\gamma}$ and therefore, $H > 0$ implies that wave function should have negative eigenvalue of $\hat{P}_\gamma$. Indeed, if we identify $P_\gamma = -i\frac{\partial}{\partial\gamma}$, then the Henkel function in the limit of $a \to 0$ gives the requisite eigenvalue. For large values of a i.e. $a \to \infty$, the behaviour is

$$H_{\frac{i}{2}\nu}^{(1)} \to \text{Const}\frac{e^{-\frac{a^2}{2}}}{a} \tag{55}$$

which is well behaved for asymptotic values of the scale factor.

The wave function (53) is in the factorized form, where the dependence on the scale factor a given by the Hankel function, $H_{\frac{i}{2}\nu}^{(1)}(\frac{i}{2}a^2)$ and the dependence on χ and ϕ (through α and β) are given by the associated Legendre polynomial $P_{-\frac{1}{2}+i\rho}^m(\cosh\alpha)$ and the phase factor $e^{im\beta}$. Notice that α and β are related to χ and ϕ through equation (36). It is not very easy to extract the behaviour of the wave function in

13

the strong or in the weak coupling limit although the behaviour of the associated Legendre polynomial is well known for small or large values of its argument $\cosh\alpha$. Therefore, it is useful to look at the solution to the axion-dilaton part of the wave function in terms of the set of variables λ_1 and λ_2 introduced in the previous section. We may recall the introduction of the complex field $\lambda = \lambda_1 + i\lambda_2$. We write

$$\lambda = \lambda_1 + i\lambda_2 \equiv x + iy \tag{56}$$

and therefore, $x = \chi$ and $y = e^{-\phi}$. The Hamiltonian, $\tilde{H}$, satisfies the constraint

$$\tilde{H} = \frac{1}{2}\left(\frac{P_a^{\;2}}{a} + a - \frac{y^2}{a^3}(P_x^2 + P_y^2)\right) = 0 \tag{57}$$

This is the classical Hamiltonian constraint. P_x and P_y refer to the canonical conjugate momenta of the coordinates (moduli) x and y respectively. In passage to WDW equation, we represent P_a, P_x and P_y as differential operators and also take into account operator ordering as alluded to above. The wave function can be expressed in the factorized form.

Let us focus on the part of the Hamiltonian depending on x and y coordinates

$$\tilde{H}(x, y) = \frac{1}{2}y^2(P_x^2 + P_y^2) \tag{58}$$

We immediately recognize that this Hamiltonian descibes motion in the Poincaré upper half plane since $-\infty \le x \le +\infty$ and $y = e^{-\phi}$ takes only positive values. The eigenvalue equation for this Hamiltonian has been studied in detail from several perspectives [45]. Let us denote the wave function as $\psi^{UP}(x, y)$ and it satisfies the eigenvalue equation

$$\tilde{H}(x, y)\psi^{UP}(x, y) = E_P\psi^{UP}(x, y) \tag{59}$$

where E_P is the eigenvalue for the Poincaré Hamiltonian. We recall that the problem at hand is analogous to motion of a particle on surface of constant negative curvature as was pointed out when we formulated the problems in terms of the coordinates v_i. We are examining the same problem now in variables x and y.

The eigenfunction and the eigenvalues are given by

$$\psi^{UP}(x, y) = \sqrt{\frac{\rho\sinh\pi\rho}{\pi^3}}\sqrt{y}K_{i\rho}(|\kappa|y)e^{i\kappa x} \tag{60}$$

and

$$E_P = \left(\rho^2 + \frac{1}{4}\right) \tag{61}$$

Here, $\rho > 0$ and zero eigenvalue for κ is excluded i.e. $\kappa \ne 0$. The wave function, $\psi^{UP}(x, y)$, satisfies the orthonormality condition

$$\int_{-\infty}^{+\infty} dx \int_0^{+\infty} \frac{dy}{y^2} \psi^{UP}_{\rho,\kappa}(x, y)\psi^{UP\star}_{\rho',\kappa'}(x, y) = \delta(\rho - \rho')\delta(\kappa - \kappa') \tag{62}$$

14

and the completeness relation is

$$\int_{-\infty}^{+\infty} d\kappa \int_{0}^{+\infty} d\rho \psi_{\rho,\kappa}^{UP}(x',y') \psi_{\rho,\kappa}^{UP\star}(x,y) = yy'\delta(x-x')\delta(y-y') \tag{63}$$

Note the appearance of yy' which is due to definition of the line element in the upper half Poincaré plane.

Let us discuss some of the features of the wave function ψ^{UP}. We recall that the Hamiltonian $\tilde{H}$ contains only P_x and has no explicit x-dependence, whereas it depends on P_y as well as on y. Thus, we expect that the wave function will have a plane wave component in x (which amounts to a plane wave in the axion field) to due the translational invariance in this coordinate. We remind the reader that the action (1) depends on axion field through its derivative and consequently, the equations of motion for χ is a conservation law. This is manifest in our solution. The y-dependent part of the wave function contains the Bessel function $K_{i\rho}(|\kappa|y)$, besides the $\sqrt{y}$ factor. Next, we look at the behaviour of the wave function in the limit $\phi \to \infty$ which corresponds to the strong coupling limit and this amounts to taking $y \to 0$ limit. The Bessel function in this limit has the behaviour $\lim_{y\to 0} \; K_\nu(y) \sim \frac{1}{2}\Gamma(\nu)(\frac{z}{2})^{-\nu}$. We remind the reader that $y = e^{-\phi}$ and $\nu = i\rho$ in our case. Thus, the Bessel function looks like a plane wave in this limit. Therefore, the wave function, $\psi^{UP}(x,y)$, behaves like a plane wave in both axion and dilaton with an additional factor of $\sqrt{y}$.

Now we consider the solution to (59) in a special case where the wave function ψ_{UP} is x-independent. We would like to examine how the wave function is related to solutions in the semiclassical approximation in this simple example. Notice that case the wave equation is the Helmholtz equation $-y^2 \frac{d^2\psi}{dy^2} = \epsilon\psi$ and the two solutions are $y^{\frac{1}{2}+i\rho}$ and $y^{\frac{1}{2}-i\rho}$ and $\epsilon = \rho^2 + \frac{1}{4}$, as before. For $\epsilon > \frac{1}{4}$, the solution

$$y^{\frac{1}{2}+i\rho} = y^{1/2} e^{i\rho\ln y} \tag{64}$$

We remark, in passing, that x-dependent solutions may be generated from the wave function (64) by implementing a suitable S-duality transformation given by

$$z \to z' = \frac{az+b}{cz+d}, \quad \text{with } ad - bc = 1 \tag{65}$$

and the imaginary part of z', y', is given by

$$y' = \frac{y}{|cz+d|^2} \tag{66}$$

Thus the transformed wave function will carry x-dependence. The following interpretation of the wave function is worth noting as we go over to the classical limit. The $\hbar \to 0$ limit corresponds to $|\rho| \to \infty$. This can be seen as follows: if we restore the all dimensionful parameters such as radius of the pseudosphere, $\mathcal{R}$, Planck's constant and the mass of the particle m_p, then we can define an energy scale (in quantum

15

mechanics) $e_s = \frac{\hbar}{2m_p\mathcal{R}^2}$. Thus energy can be expressed as $E = e_s\zeta$ in terms a dimensionless 'eigenvalue',ζ. Therefore, the semiclassical limit is equivalent to sending ζ to ∞. This is also same as sending ρ to asymptotic values. From semiclassical arguments (say like WKB)

$$\psi \sim Ae^{iS} \tag{67}$$

We can identify $\ln y$ as the classical action since it is $\int_1^y ds$ along the y axis for the case at hand i.e. $x = 0$ is chosen. Moreover, $\sqrt{y}$ is the invariant density in the Poincaré metric. Therefore, we can identify $\psi(y)$ as the solution in the WKB approximation. Let us write down the full wave function as a solution to WDW equation in terms of the scale factor, a and the Poincaré coordinates x and y:

$$\Psi(a, x, y) = H_{\frac{i}{2}\nu}^{(1)}(\frac{i}{2}a^2)\psi^{UP}(x, y) \tag{68}$$

This wave function is well behaved in the strong coupling limit. If we look at the classical solution to axion-dilaton cosmological solution, the dilaton approaches strong coupling regime as t approaches zero and in that limit the scale factor tends to zero. We should keep in mind, however, that the tree level string effective is not quite reliable in this domain and it might be necessary to add higher order terms in curvatures and additional stringy correction terms to the tree level effective action. One possible way to deal with solution at the strong coupling domain is to apply an S-duality transformation so that we go over to the weak coupling regime. This might be good strategy to adopt since we are dealing with S-duality invariant action. It is important to note that we have succeeded in presenting an exact and complete solution to WDW equation for axion-dilaton string cosmology which was lacking so far.

4   Hidden Symmetry in Axion-Dilaton Cosmology

We have argued that the axion-dilaton string cosmology might be endowed with higher symmetry since the string effective actions in lower dimensions exhibit enhanced symmetries. The string effective action depends only on the cosmic time coordinate in the cosmological models. In our previous note [8], we presented evidence for existence of a w_∞ algebra in the form proposed by Bakas [29]. Our result was based for a specific choice of the Casimir, $C = -\frac{3}{16}$ of the S-duality group $SU(1,1)$. It was argued that unitary representations of $SU(1,1)$ are infinite dimensional. Thus the wave function associated with the WDW equation for axion, dilaton and graviton is highly degenerate. Moreover, the generators of $SU(1,1)$ commute with the action and hence with the Hamiltonian due to S-duality invariance of the theory. For the specific choice of the Casimir, $(C = -\frac{3}{16})$, we expressed the raising, lowering generators $J_\pm$ and the compact diagonal generator J_3 in terms of the creation and annihilation

operator of a single boson. We recall that

$$J_+ = \frac{1}{2}(a^\dagger)^2, \ J_- = \frac{1}{2}a^2, \ J_3 = \frac{1}{2}(a^\dagger a + 1/2) \tag{69}$$

$[a, a^\dagger] = 1$. Defining, $|n> = (n!)^{-1/2}(a^\dagger)^n|0>$, where $a|0>= 0$ is the condition on vacuum, we find

$$J_+|n> = \frac{\sqrt{(n+1)(n+2)}}{2}|n+2>, \tag{70}$$

$$J_-|n> = \frac{\sqrt{(n(n-1)}}{2}|n-2>, \tag{71}$$

and

$$J_3|n> = (n + \frac{1}{2})|n> \tag{72}$$

This representation for $J_\pm$ and J_3 is due to Holman, Biedenharn and Ui [30]. In this case, we get two different representations of the Lie algebra from $|n>$: (i) For odd n, $j = -3/4$ and (ii) for even n, $j = -1/4$ which belongs to the supplementary series. When J_3 and $J_\pm$ are defined as in (69), a suitable set of operators can be constructed, in both the cases, which satisfy the w_∞ algebra [29]

$$[\mathcal{V}_m^{(k)}, \mathcal{V}_n^{(l)}] = ((k+1)(n+1) - (m+1)(l+1))\mathcal{V}_{m+n}^{(k+l)} \tag{73}$$

where $\mathcal{V}_m^{(k)} = (a^\dagger)^{m+1}a^{k+1}$, $n, m \geq -1$. Here, we have computed the classical algebra, ignoring normal orderings.

The presence of a w_∞ in axion-dilaton cosmology was unveiled in a heuristic manner for a special choice of the Casimir so that the generators of $SU(1,1)$ have a very simple representation in terms of a single boson operator. In recent years w_∞ and W_∞, which might be considered as deformation of the former, have played useful roles in a variety of problems in physics. They first appeared as $N \to \infty$ limit of W_N algebras [31]. These algebras were studied in the context of $c = 1$ theories [32], in large N limit of $SU(N)$gauge theories [33] and in quantum Hall effect [34]. In what follows, we shall summarize some of the essential aspects of W-infinity algebras and then argue that axion-dilaton cosmology also shares those features. These algebras may be conceived as $N \to \infty$ limit of W_N algebras. The W_N algebras are extended Virasoro algebras with a field content of higher conformal spins: $2, 3, ...N$. There are nonlinear terms in the commutation relations and such terms get more and more complicated as N takes higher and higher values. Therefore, these algebras do not correspond to Lie algebras so long as N is finite. When the limit $N \to \infty$ is taken, it is argued that the algebra assumes a rather simple form [29]. Of course, the passage

to $N \to \infty$ limit is not unique; and consequently, different algebras emerge depending on how one first rescales the generators and the structure constants of W_N algebras. Bakas proposed a prescription for the $N \to \infty$ limit to arrive at an algebra, known as w_∞

$$\left[w_m^{(i)}, w_n^{(j)} \right] = [(j-1)m - (i-1)n]\, w_{m+n}^{(i+j-2)} \tag{74}$$

Note that $w_m^{(i)}$ are generators of conformal spin i. In this case the algebra has central charge *only* for $i = 2$ which turns out to be the Virasoro algebra. On the other hand, in the case of W_N algebra, there is central term for each of the conformal spin. Pope, Romans and Shen [35] proposed the W_∞ algebra which may be interpreted as the deformed w_∞ algebra which has central term for all conformal spins. The Jacobi identity is imposed to derive the structure constants. They show that W_∞ algebra has a subalgebra whose generators are conformal spins $2, 3, 4, ...$ in the $\underline{3}$, $\underline{5}$, $\underline{7}$... representations of $SL(2, R)$. The resulting algebras are anomaly free. One constructs tensor operators of $SL(2, R)$, $\mathcal{T}(C)$, where C is the value of the Casimir generator of $SL(2, R)$. In order to study the subalgebras of the W_∞ algebra, called wedge algebras $W_\wedge$, the generators $V_m^{(i)}, |m| \le i+1$ are identified. The commutators of the generators belonging to this subset gives another generator which also belongs to the same subset. The $W_\wedge$ is anomaly free. The generators belonging to $W_\wedge$ are constructed as family of tensor operators which are in $SL(2, R)$ satisfying special properties. The starting point is to consider the three generators of $SL(2, R)$, $\{L_\pm, L_0\}$, which transform as $\underline{3}$. We consider higher tensor operators T_m^l, $-l \le m \le l$ transforming as the $(2l+1)$-dimensional representations of $SL(2, R)$. Such tensors are constructed from the polynomials of degree l in the generators $L_\pm$ and L_0. One of the prescription is to start from the highest weight state $T_l^l \equiv (L_+)^l$. We act with the step down operator L_- successively in order to construct lower weight states. After $(l-m)$ such operations, as specified below, one arrives at

$$T_m^l = \frac{1}{(-2l)_{l-m}} \left(Ad_{L_-} \right)^{l-m} (L_+)^l \tag{75}$$

where $Ad_X(Y) = [X, Y]$; therefore (75) amounts to taking $(l - m)$ commutators of L_- with $(L_+)^l$ i.e. $[L_-, [L_-, [L_-...(L_+)^l]]]]]$. The denominator $(-2l)_{l-m}$ has the meaning $(a)_n = \frac{(a+n-1)!}{(a-1)!}$. We remind the reader that $[L_\pm, L_0] = \pm L_\pm$ and $[L_+, L_-] = 2L_0$. Therefore, we can expand $T_{\pm m}^l$ as polynomials in L_0 times $(L_\pm)^m$ which have coefficient expressed in terms of the Casimir of $SL(2, R)$.

$$T_{\pm m}^l = \left(L_0^{l-m} \pm \frac{1}{2}(l-m)L_0^{l-m-1} + ... \right) L_\pm^m \tag{76}$$

The normalization is so adjusted that the first term has coefficient unity. As is the case with $SU(1, 1)$, $SL(2, R)$ has infinite dimensional unitary representations. For a given choice of the value of the Casimir, say C, T_m^l, the set of tensor operators

18

constructed above, close into an infinite dimensional algebra labeled as $\mathcal{T}(C)$. It is argued that the wedge algebra coincides with $\mathcal{T}(C)$ for some appropriate value of C. One finds that the wedge algebra contained in W_∞ is $SL(2, R)$ operator algebra $\mathcal{T}(C = 0)$; the Casimir takes zero value. Arguments for the connection between wedge algebra and the $SL(2, R)$ operator algebra due to Pope, Romans and Shen [35] are based on abstract constructions. Since $SL(2, R) \sim SU(1, 1)$, the above argument will go through for our cosmological problem. In fact, we could have adopted $SL(2, R)$ as our S-duality group as well. Notice that we have realizations of the generators (of $SU(1, 1)$)

$$J_\pm = \mp e^{\pm i\beta}\frac{\partial}{\partial\alpha} - i\coth\alpha \; e^{\pm i\beta}\frac{\partial}{\partial\beta}, \quad J_3 = -i\frac{\partial}{\partial\beta} \tag{77}$$

in terms of the coordinates of the pseudosphere (in turn the axion and the dilaton). Therefore, the set of operators T_m^l can be constructed from the generators (77). We recall that our eigenfunctions correspond to the continuous series representation of $SU(1, 1)$. Consequently, when we fix the value of the Casimir, the magnetic quantum number takes values, $m = 0, \pm 1, \pm 2, \pm 3, \ldots$. as mentioned earlier. We can construct the operators $T_{\pm m}^l$ according to our convenience and the algebra will be realized on the infinitely degenerate eigenstates. Furthermore, the exact eigenfunctions have also been constructed by us in this case. We are led to the conclusion that in this case we have a physical realization of the algebra proposed by Pope, Romans and Shen [35]. We would like to discuss a limit when the Casimir, $C \to \infty$. This corresponds to the large eigenvalue of the quantum mechanical equation on the pseudosphere. This limit is of relevance due to the fact that the semiclassical approximation can be taken in the large C regime. In study of quantum mechanics on the pseudosphere, the inverse of the eigenvalue corresponds to $\hbar^2$ and $\hbar \to 0$ (the semiclassical limit) amounts to studying the system for large eigenvalues. It is important in the context of the study the chaotic properties of the motion on the pseudosphere as we shall briefly discuss in the following section. Furthermore, the algebra of the operators in large C limit also takes rather simple form. If we define the set $\{\hat{T}_m^{(i)}\}$ to be the suitably rescaled generators from the set $\{T_m^{(i)}\}$, then the algebra becomes

$$\left[\hat{T}_m^{(i)}, \hat{T}_n^{(j)}\right] = \sum_k \frac{\sqrt{2k+1}(i+j+k+2)\Delta^2(\frac{i}{2}, \frac{j}{2}, \frac{k}{2})}{\Delta(i, j, k)} C_{m,n,m+n}^{ijk}\hat{T}_{m+n}^{(k)} \tag{78}$$

where

$$\Delta(a, b, c) \equiv \sqrt{\frac{(a+b-c)!(b+c-a)!(c+a-b)!}{(a+b+c+1)!}} \tag{79}$$

and C_{mnp}^{ijk} are Clebsch-Gordan coefficients whose properties can be found in ref. [36]. In (78), the sum over k is over all values of k with the constraint that $i + j + k$ odd and the Clebsch-Gordan coefficient $C_{m,n,m+n}^{ijk}$ have nonzero values. Notice the absence any central term in the algebra (78). As we have argued, large eigenvalue limit is the

19

semiclassical limit and therefore, central terms, if any, are not expected to make their appearances in these algebras.

It is worth while to mention that the novel symmetry, associated with cosmological scenario of axion, dilaton and graviton, owes its origin to the existence of the S-duality group $SU(1,1) \sim SL(2,R)$. The highly degenerate wave function in the axion-dilaton sector exists due to the same reason.

5 Axion-Dilaton Cosmology and Chaos

In this short section, we briefly discuss chaos in axion-dilaton cosmology. This is another interesting aspect. The axion-dilaton moduli parametrize a space of constant negative curvature as is evident from its parametrization as the surface of a pseudo-sphere or its parametrization as Poincaré upper half plane. Motion on such a surface is known to be chaotic. However, in the cosmological scenario the moduli are coupled to gravity; in our case it is the FRW metric.

The study of chaos, in the context of cosmology, started more than three decades ago [37, 38] and an early review is contained in the article by Barrow [39]. Misner [40] and Chitre [41] studied chaos in Bianchi type VIII and IX cosmologies. When one is sufficiently close to the singularity, in a suitable coordinate system,the problem can be reduced to a special two dimensional billiard on a homogeneous space of constant negative curvature. Recently, the billiard problem for Einstein-dilaton gravity along with p-form fields has been studied in detail and we refer the reader to the article of Damour, Henneaux and Nicolai [42].

Let us turn our attention to string theory. In string theory, we encounter moduli spaces where the generic characteristics is such that a noncompact group is modded out by its maximal compact subgroup and eventually realized as a discrete subgroup. The well known example is compactification of a bosonic string [43] (we illustrate it as a simple case) on a d-dimensional torus, T^d. The moduli, $\mathcal{M}$ parametrize $\frac{O(d,d,Z)}{O(d) \otimes O(d)}$. Thus the effective action can be expressed as nonlinear σ-model in the target space and is invariant under global $O(d,d,Z)$ and local $O(d) \otimes O(d)$. From the point of view of the evolution of the string, we usually encounter situations when a discrete group Γ connects different configurations of the spacetime background field in the action and then the sigma models correspond to same conformal field theory. The S-duality group is $\frac{SL(2,R)}{U(1)}$ (we remind that $SL(2,R) \sim SU(1,1)$) and eventually $SL(2,R)$ is reduced to $SL(2,Z)$ in full quantum theory. Moreover, as mentioned in the introduction, vacuum expectation values of the dilaton and the axion are related to the coupling constants and the VEV of the axion is related to the θ-angle that appears as coefficient of the topological term of gauge theories in the four dimensional action. The solution to equations of motion in Poincaré half plane is well understood. The evolution of $\lambda \equiv x + iy$ corresponds to semicircular trajectories lying in the upper half Poincaré plane. The center of the semicircles lie on the real axis; the location

20

of the center and radius of the semicircles are determined from the initial conditions supplied to solve the classical equations of motion. In the cosmological scenario, as discussed in section 2, we aim at solving coupled set of equations involving metric and the matter fields. When we consider axion-dilaton evolution in the cosmological context (with FRW metric), the equation of motion (30) is modified due the presence of the Hubble parameter, H, which couples to $\dot{\xi}$. Thus it is like a frictional force - this is already well known from the evolution equation of the scalar field, for example in models of inflation.

Let us briefly recapitulate the situation for the classical dynamics and quantum equations of motion in our problem (motion on the pseudosphere): (i) classical equations are coupled differential equations. (ii) The WDW equation is solved by adopting the method of separation of variables and we have a quantum mechanical eigenvalue equation for axion-dilaton. Horne and Moore [44] persuasively argue that classical chaotic behaviour continues to be exhibited by axion-dilaton in the presence of gravity - when the frictional force of Hubble parameter is present. Usually, the classical chaotic motion is characterized by two attributes: (a) The phase space trajectories diverge exponentially and (b) the admissible phase space volume be compact. It is important to note that when the motion is unrestricted on the pseudosphere, the trajectories still diverge. In order to satisfy the periodicity properties of the trajectories/orbit one has to construct the compact surfaces the through introduction of the notion of suitable periodic boundary conditions in this curved space; just as we construct a torus by identifying opposite sides of a rectangle in a two dimensional Euclidean plane. This is the way to get finite volume phase space.

In the quantum mechanical considerations, we do not discuss particle trajectories/orbits since we solve for wave functions. The connection with chaotic behaviour may be established in the semiclassical regime. In view of the above discussions, we may argue that semiclassical solutions to the wave equation on unrestricted pseudosphere might be able to bring out one of the features of chaos. When we solved for the wave equation in terms of variables α and β, we chose the wave function to be periodic in β since the compact generator J_3 was required to be diagonal. The large eigenvalue limit is the classical limit where $\rho \rightarrow$ takes asymptotic values. Consequently, the wave function, associated Legendre polynomial is to be evaluated in large ρ limit.

Now, we proceed to discuss solutions to wave equation in the semiclassical approximation and its connection with the chaotic behaviour of classical theory. We have shown in section 3 that plane wave solutions are ($E_U \sim \rho^2$ for large ρ)

$$\psi_U \sim \sqrt{y}e^{\pm i\sqrt{E_U}\log y} \tag{80}$$

This solution is expressed in terms of the coordinates of Poincaré upper half plane. These plane waves are WKB solutions [12, 45]. Therefore, we may identify the action $S = \pm\sqrt{E_U}\log y$ and $A^2(S) = y = e^{-\frac{S}{\sqrt{E_U}}}$; here S being the action that appears in the exponential when we derive the WKB wave function and $A(S)$ is the coefficient of the exponential. If we define $\tau = \frac{S}{2E_U}$, then, $A^2(\tau) = e^{-\sqrt{2E_U}\tau}$. $A^{-2}(S)$ measures

21

divergence of trajectories in the semiclassical approach to understand chaos in the surface of constant negative curvature (Poincaré upper half plane) [12]. Thus one extracts the Liapunov exponent by examining large S behaviour of $A^{-2}(S)$ In our convention, E_U carries dimension of inverse time and is related to the Liapunov exponent ω. If we are to compare with the convention followed in literature for the study of chaos [12, 45] for the exponent ω, we should set $E_U = 2$ which gives $\omega = 2$. This is the value obtained for Liapunov exponent for exponential divergence of classical trajectories in the study of chaos in Poincaré upper half plane [12, 45]. Thus the WKB approximation reproduces one of the features of chaos in the classical dynamics. We would like to remind the reader that the correspondence is for chaos of x, y coordinates when motion is in flat Minkowski background. However, it is not obvious, although axion-dilaton cosmology is argued to be chaotic [44], that the classical trajectories will diverge with the same exponent as the flat spacetime case. Nevertheless, it is quite encouraging that the quantum cosmological solutions, in the semiclassical approximation, does bring out one of the features of chaos.

We end this section with following comments. In order to study chaos on the pseudosphere, we have to introduce periodic boundary conditions. This is achieved following a set of prescriptions as discussed below. An immediate consequence of the periodic boundary condition on the compact surface, constructed from pseudosphere, for the quantum theory is that the energy eigenvalue becomes discrete. Moreover, for the unconstrained pseudosphere, the energy eigenvalue and wave functions can be solved exactly. The compact surface with negative constant curvature, obtained from the pseudosphere, corresponds to a Riemann surface of at least genus two, $g = 2$, although higher genus Riemann surfaces can give rise to constant negative curvature surfaces. It is a formidable task to solve for eigenfunction and eigenvalues of the Hamiltonian which satisfy ' periodic' boundary conditions [12, 45]. There are no exact straight forward solutions to the general problem. The problem is further complicated in the cosmological context since one has to account for the presence of gravity while addressing the question of chaotic phenomena. We feel that considerable amount of efforts are necessary to provide a complete understanding of chaos in axion-dilaton cosmology.

6   Summary and Discussions

We have investigated various aspects of axion-dilaton string cosmology in this article. As a first step, before addressing the cosmological problem, we studied the S-duality properties of string effective action in four spacetime dimensions and constructed (classical) Nöther currents and corresponding charges which are the generator of S-duality group. Axion and dilaton parametrize the coset $\frac{SU(1,1)}{U(1)}$. We considered isotropic, homogeneous, FRW metric as our cosmological model. The moduli depend only on the cosmic time coordinate, t. When one solves for the classical equations

of motion, there is curvature singularity, corresponding to bing bang and the dilton goes to ∞ in this limit. Such a feature of the solutions is observed for closed, open and flat Universe. It was shown that axion-dilaton parametrize the surface of a pseudosphere in a suitably defined coordinate system. Therefore, the action, in the cosmological context for axion and dilaton, can be cast in a form which is analogous to the Lagrangian of a particle moving on the surface of a pseudosphere in the presence of FRW spacetime metric.

The Wheeler De Witt equation allows separation of the variables and consequently, the wave function is factorisable as a product of two functions; one depending on the scale factor and the other on the moduli. The S-duality invariance of the action leads to the Hamiltonian which is sum of two pieces; one which depends only on the Casimir of the S-duality group and a second part corresponds to the Ricci scalar, computed in the Einstein frame metric. Thus, we can solve for the wave function of the moduli from the study of the representation theory of the S-duality group. The solutions correspond to the continuous representation of $S(1,1)$. Therefore, the solution of the WDW equation, i.e. $\hat{\mathcal{H}}\Psi = 0$ is infinitely degenerate. In other words, the zero eigenvalue solution to wave equation has infinite degeneracy. This is a consequence of the fact that unitary representations of noncompact groups are infinite dimensional. It is believed that the exact symmetry is the discrete group $SU(1,1,Z) \sim SL(2,Z)$. We have solved WDW equation without introducing a potential for the axion and dilaton. A potential for the moduli could be generated due to nonperturbative effects such as gluino condensation, in the cosmological action. Horne and Moore have put forward proposals to construct such a potential on general grounds [44]. They impose some constraints in deriving the form of the potential: that it respects discrete S-duality symmetry and perturbation theory be valid in the weak coupling regime. It is quite plausible that the gluino condensation occurs much below the regime of quantum cosmology. In this respect, we may justify our solution of WDW equation in absence of a potential for the moduli. Nevertheless, it is desirable to look for a mechanism which lifts the degeneracy of the wave function so that one obtains a unique ground state solution.

We have provided solutions to WDW equation in another set of coordinates where the moduli define the upper half complex plane - the Poincaré coordinates. This choice has two interesting features. First, behavior of the wave function of the moduli, when dilaton takes positive asymptotic values can be studied in a simple manner. This is the strong coupling limit of the theory. Moreover, as we have argued earlier, when the scale factor tends to zero (i.e. $a(t) \to 0$ with $t \to 0$) in the classical solution, $\phi \to \infty$ in the same limit. Thus, the behavior of the wave function near the curvature singularity (of classical solution) is rendered transparent. The second feature is that the semiclassical approximation could be achieved efficiently since the large eigenvalue limit is identified as the WKB limit and the corresponding action can be read off immediately from the form of the wave function in the Poincaré coordinates.

We have proposed [8] that there is an observed tendency that the lower dimensional

string effective actions are endowed with enhanced symmetries and the symmetry groups are enlarged as we go to lower and lower dimensions. The cosmological axion-dilaton action depends only on a single coordinate - the cosmic time. We argued that there is an underlying w_∞ algebra for a specific choice of the Casimir [8]. In this article we provide a construction of $W_\wedge$ algebra following the prescriptions of ref.[35], It is shown that the algebra assumes a simple form, without central terms, when the Casimir takes asymptotic values; which is the semiclassical limit as alluded to earlier. We find the discovery of W-infinity algebra in the context of axion-dilaton cosmology to be one of the novel outcomes of our investigation. It is worth while to point out that the presence of large exceptional groups and appearance of Kac-Moody algebras have been noticed in the context of cosmological billiards [42] and this result is another evidence in favour our conjecture that lower dimensional effective actions possess higher symmetries.

The 5th section was devoted to study chaos in axion-dilaton cosmology. We showed, from qualitative arguments, that the system exhibits chaos in the semiclassical limit. The semiclassical limit amounts to determining wave functions for large eigenvalues. This is same as taking $\hbar \to 0$ limit. We are encouraged to find the supportive evidence in favour of chaos from our semiclassical analysis. Classical axion-dilaton cosmology is known to be chaotic as has been argued in ref. [44]. We are aware that a rigorous study of chaos in quantum axion-dilaton cosmology will require resolution of many difficult mathematical problems and the results presented here are qualitative. We feel that our preliminary results and our attempts to study chaos in quantum string cosmology will initiate further investigations in these directions.

7 Appendix

We summarize some of the properties of $SU(1,1)$ as useful information. This is group of 2-dimensional, unimodular matrices and the Lie algebra of its generators $\{J_1, J_2, J_3\}$ is given by (10)

$$[J_1, J_2] = -iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2 \tag{81}$$

and the Casimir is $-J_1^2 - J_2^2 + J_3^2$. We have already given the explicit realization of the generators in terms of the Pauli matrices in section 2 and those generators generate nonunitary representations of $SU(1,1)$. In order to generate unitary representations of the group; we have to provide a realization of the Lie algebra in terms of Hermitian operators. Thus, we should have an infinite dimensional vector space, because all irreducible representations of noncompact Lie groups are infinite dimensional. Let us start with unitary representation of $SU(1,1)$ in a Hilbert space $\mathcal{H}$ and it will be infinite dimensional. The matrix $S = \begin{pmatrix} \alpha & \beta \\ \beta^\star & \alpha^\star \end{pmatrix} \in SU(1,1)$. This has a maximal compact group $U(\delta) = \begin{pmatrix} e^{i\frac{\delta}{2}} & 0 \\ 0 & e^{-i\frac{\delta}{2}} \end{pmatrix}$. Now $\mathcal{H}$ can be decomposed with respect to $U(\delta)$ into direct sum of one dimensional representation $|m>$.

$$\mathcal{H} = \oplus|m>, \quad U(\delta)|m> = e^{im\delta}|m>, \quad |m> \in \mathcal{H} \tag{82}$$

$\delta = 0 \mod 4\pi$ is the identity of the group; $m \in \mathbf{Z}$, or $m \in \mathbf{Z} + \frac{1}{2}$. Note that quantization of m does not follow from the Lie algebra (unlike the compact case), but from the global properties. In decomposition $\mathcal{H} = \oplus|m>$ each m appears only once, since the Casimir and J_3 form a complete system of commuting observables.

In section 3, we have briefly discussed the classification of the representations of $SU(1,1)$. The representations come in two types depending on the choice of magnetic quantum numbers: (i) Integral magnetic quantum number, and (ii) half integer quantum number. There are four series: (a) Finite dimensional nonunitary representations. (b) Positive discrete series of infinite unitary representations, D_j^+. (c) Negative discrete series, D_j^- and (d) Continuous series of unitary, representations, C_j.

When we consider rotations in three dimensional Euclidean space, three Euler angles are introduced for any arbitrary rotation, since the rotation group possesses three compact generators. Similarly, in the case of $SU(1,1)$ one introduces three 'angles'. We can write a matrix element as $< j, m'|e^{iJ_3\phi_1}e^{iJ_2\eta}e^{iJ_3\phi_2}|jm >$, where ϕ_1, η, ϕ_2 are the Euler angles. The infinite dimensional representations are given below:

$$D_{mn}^{j+}(a) = \Theta_{mn}(j)\alpha^{\star-(m+n)}b^{m-n}{}_2F_1(-n-j, 1-n+j|1+m-n| - |b|^2) \tag{83}$$

with the constraint $m, n \geq -j > 0, \quad m \geq n$

$$D_{mn}^{j+}(a) = (-1)^{n-m}\Theta_{nm}(j)\alpha^{m+n}b^{\star m-n}{}_2F_1(-m-j, 1-m+j|1+n-m| - |b|^2) \tag{84}$$

25

with the condition that $m, n \geq -j > 0, \quad n \geq m$

$$\Theta_{mn}(j) = \frac{1}{(m-n)!} \left[\frac{(m+j)!(m-j-1)!}{(n+j)!(n-j-1)!} \right]^{\frac{1}{2}} \tag{85}$$

$$D_{mn}^{j-}(a) = \Theta_{mn}(j)\alpha^{*m+n}b^{m-n}{}_2F_1(-m-j, 1+m+j|1+m-n| - |b|^2) \tag{86}$$

Now the constraints, for the negative discrete series are $m, n < j < 0, \quad m \geq n$

$$D_{mn}^{j-}(a) = (-1)^{n-m}\Theta_{nm}(j)\alpha^{m+n}b^{\star m-n}{}_2F_1(n-j, 1+n+j|1+n-m| - |b|^2) \tag{87}$$

and the conditions to be satisfied are $m, n < j < 0, \quad n \geq m$

$$\Theta_{mn}(j) = \frac{1}{(m-n)!} \left[\frac{(-n+j)!(-n-j-1)!}{(-m+j)!(-m-j-1)!} \right]^{\frac{1}{2}} \tag{88}$$

for the D_{mn}^{j-} series. The parameters appearing in these functions are defined below,

$$j = -\frac{k}{2}, \quad k = 1, 2, 3.... \tag{89}$$

$$\alpha = e^{-i\phi_1}\cosh\frac{\eta}{2}e^{-i\phi_2}, \quad \text{and} \quad b = e^{-i\phi_1}\sinh\frac{\eta}{2}e^{i\phi_2} \tag{90}$$

a appearing in the argument collectively stands for three Euler angles. The Casimir is $-j(j+1), \ j < 0$.
Now we define the relevant C-functions for the continuous series

$$C_{mn}^j(a) = \Theta_{nm}(j)\alpha^{\star m+n}b^{m-n}{}_2F_1(\frac{1}{2} + m + \sigma, \frac{1}{2} + m - \sigma|1+m-n| - |b|^2) \tag{91}$$

and $m \geq n$

$$C_{mn}^j(a) = \Theta_{nm}(j)\alpha^{m+n}b^{\star m-n}{}_2F_1(\frac{1}{2} + n + \sigma, \frac{1}{2} + n - \sigma|1+n-m| - |b|^2) \tag{92}$$

here $n \geq m$

$$\Theta_{mn}(j) = \frac{1}{(m-n)!}\Pi_{k=1}^{m-n} \left[\frac{1}{4} - \sigma^2 + (n+k)(n+k-1) \right]^{\frac{1}{2}}, \quad m > n \tag{93}$$

and

$$\Theta_{mn}(j) = \frac{(-1)^{n-m}}{(m-n)!}\Pi_{k=1}^{n-m} \left[\frac{1}{4} - \sigma^2 + (m+k)(m+k-1) \right]^{\frac{1}{2}}, \quad n > m \tag{94}$$

26

The Casimir $j(j + 1) \equiv \frac{1}{4} - \sigma^2 = \frac{1}{4} + s^2 = q$. For the continuous series: Re $j = -\frac{1}{2}$, $0 < \text{Im } j < \infty$ or Im $j = 0$, $-\frac{1}{2} < \text{Re } j < 0$ The orthogonality relation for the discrete series is

$$\int da \ D_{mn}^{j\pm}(a) D_{m'n'}^{\star}{}^{j'_{\pm}} = -\frac{\delta_{mm'}\delta_{nn'}\delta_{jj'}}{2j + 1} \tag{95}$$

Moreover,

$$\int da \ D_{mn}^{j\pm}(a) D_{m'n'}^{\star}{}^{j'_{\mp}} = 0 \tag{96}$$

For the continuous series, Re $j = -\frac{1}{2}$ and $o < \text{Im } j < \infty$, the relevant orthonormality relation is

$$\int da T_{mn}^{q}(a) T_{m'n'}^{q'}{}^{*}(a) = \frac{2\pi \delta_{mm'}\delta_{nn'}\delta(s - s')|\Gamma(2is)|^2}{|\Gamma(\frac{1}{2} - n - is)\Gamma(\frac{1}{2} + n + is)|^2} \tag{97}$$

The measure of the integration is given by

$$\int da = \frac{1}{2(2\pi)^2} \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \int_0^{\infty} d\eta \ \sinh \eta \tag{98}$$

It follows from a fundamental theorem in harmonic analysis with $SU(1, 1)$ that we can bring together the two types of solutions: discrete and continuous series discussed above. The completeness property can be used to expand any square integrable function of the rotation angle, η, in terms of the continuous series and finite number of representations of the discrete series (see Rühl for example [22]).

27

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